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ABSTRACT

This paper presents an alternate perspective for utilizing the action/process/object framework when discussing student conceptions of function. This perspective extends previously used frameworks by incorporating student conceptions that deal with functional properties and situational contexts, but differs in the analysis of the nature of the functional properties that are considered. The hypothesis is that students not only use notions of set-based correspondences and transformations to think of functions as mathematical objects, but functions are also conceived as mathematical objects that possess (or fail to possess) growth properties of specific functions. These properties include symmetry, linearity, and continuity. It is argued that all students who develop a structural view utilize functional properties, but a focus on different kinds of properties produce different conceptions of function. (Author)

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A Growth-Oriented Route to the Reification of Function

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A GROWTH-ORIENTED ROUTE TO THE REIFICATION OF FUNCTION

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This paper presents an alternate perspective for utilizing the action/process/object framework when discussing student conceptions of function. This perspective extends previously used frameworks by incorporating student conceptions that deal with functional properties and situational contexts, but differs in the analysis of the nature of the functional properties that are considered. The hypothesis is that students not only use notions of set-based correspondences and transformations to think of functions as mathematical objects, but functions are also conceived as mathematical objects that possess (or fail to possess) growth properties of specific functions. These properties include symmetry, linearity, continuity, etc. It is argued that all students who develop a structural view utilize functional properties, but a focus on different kinds of properties produce different conceptions of function. Empirical evidence is used to support the theory.

Theories Of Student Conceptions Of Function

> I am astonished when I see some of the analyses which purport to be the scientific foundations on which school curriculums are to be built and find no mention of these general ideas of order and arrangement and precision. I am told that the school should teach children how to make change and how to measure wall paper and how to tell time and that sections of arithmetic should be devoted to these specific tasks, but I look in vain for any appreciation of the fact that the school ought to lead pupils who have only a hazy and unsystematic notion of the world to see the value of arrangement and order in all thinking and to cultivate the general ideas of regularity and precision.
>
> Charles Judd, 1928

Operational Views of Function

The action/process/object theory of conceptual development has received considerable attention in recent studies of functional understanding. Evidence exists that students initially acquire an *action* or *operational* view of function (Briedenbach et al., 1992; Sfard, 1989). This involves function as an operation that exists *dynamically*. An action view involves understandings pertaining to the computational aspects associated with functions, such as an arithmetic process or a "function machine." It is important to note that an action view, by definition, is concerned only with local functional properties. It is only when one can see relationships between sets of inputs and outputs that a more structural view of function can occur. But making relationships between sets of input-output ordered pairs is not an easy task (Sfard and Linchevski, 1994). The problem is compounded when functional symbolisms are confronted in very different forms (such as graphs and

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equations) and when functional ideas are confronted through purely mathematical symbols that lack a contextual basis (Filloy and Rojano, 1985). For example, when a student is asked to factor a difference of squares, what kinds of functional understandings are being built? What structural understandings of function can be acquired when a student solves a linear equation of the form $ax + b = 0$? In the latter example, if the student is allowed an opportunity to reflect on the solution, what can be acquired is a relationship between the x- and y-values of the x-intercept of the function. While reflections on this relationship support the action view of function, they can do little to promote more structural views of the concept unless this relationship is seen as a specific, local property that the entire function possesses. Otherwise, intercepts can become local phenomena of a function or set of functions, providing very specific information about one aspect of growth behavior or situational context.

Structural Views of Function

I will define a structural view of function as an extension of the action view. This will first be done by reexamining existing theories, especially those of Sfard and Dubinsky. An alternate theory for a structural view of function will then be presented that, while borrowing heavily from previous ideas, discusses student development in a different context. Sfard (1991, Sfard and Linchevski, 1994) defines a structural conception as the reification of an operational view. For example, considering the expression $3(x+5) + 1$ as a certain number rather than a computational process illustrates the beginnings of a structural view since the expression is considered to be a fixed value of an unknown. Generalizing this conception to involve the notion of variable, where the above expression simultaneously denotes several processes, represents thinking in line with functional algebra and is considered to be a structural conception.

My use of the word structural incorporates this perspective, but also includes a different domain of student thinking as well as a different context for the use of the phrase "functional property." This different context deals less with direct relationships between input-output pairs and more with specific growth properties that functions can possess or fail to possess. As will be illustrated, the connections between the two different perspectives lie in the fact that both involve understandings of functional properties. Further, some functional properties (e.g., intercepts) are tied to individual input-output pairs, while other functional properties (e.g., monotonicity, symmetry) are tied to more global views of function that relate to numerous input-output pairs. Other properties are harder to classify (e.g., continuity). Hence, the two structural views outlined below are not meant as contrasting or opposing viewpoints, but rather complementary ways of thinking about the concept of function as a mathematical object.

Relational conception of function. Sfard (1989; Sfard and Linchevski, 1994) and Dubinsky (Briedenbach et. al., 1992) describe routes through which students may *reify* the function concept (Dubinsky uses the term *encapsulate* to denote the case when students develop object-oriented conceptions of function). I will define

relational conceptions as those which deal with relationships between sets of inputs and outputs, conceptions in line with Sfard's previously discussed use of the term *structural*. Briedenbach et al. (1992) provide evidence to suggest that instruction which utilizes computer programming can promote relational conceptions. They define a *process conception* of function as a complete understanding of a given transformational activity which can be performed on a function. Students are more able to comprehend notions such as 1-1, onto, and invertibility once a process conception is achieved. They state that the process conception is a precursor to an object-oriented view of function. Several people, including Confrey and Smith (1995), argue that most textbooks and curricula support the development of function as a correspondence relation rather than as a covariance relation.

Growth conception of function. A second view of function beyond an action conception involves an understanding of functional properties specific to different kinds of functions. In previous writings I have referred to this conception as a *growth-oriented view*, but I have changed the terminology to highlight the connections between the two object-oriented conceptions that are described here. Briefly, a growth conception of function deals with the gradual awareness of specific functional growth properties of a local and global nature, followed by the ability to recognize and analyze functions by identifying the presence or absence of these growth behaviors. Global properties include symmetry and periodicity, and local properties include intercepts and points of inflection. There is evidence (Slavit, 1994) that the multi-representational capabilities of graphing technologies can lead students to develop a growth-oriented approach to the reification of function. Through an intensive look at elementary functions, students may understand function to be a related set of procedures and functional properties in a variety of functional representations. In essence, the procedures performed on functions give rise to an understanding of functional properties, other than those more familiar with the relational view (e.g., invertibility). These properties can be specific to a function class (such as linear slope) or generalize to several function classes (such as symmetry). A student can then conceive of function as an object possessing or not possessing these properties (Slavit, 1994). Particular contexts elicited by the semantic domain of the functional situation can also enrich the meanings in the conceived functional properties.

Prior work in this area (Slavit, 1994; Ruthven, 1990) has led me to hypothesize that the growth-oriented view develops in three stages. First, the growth-oriented view involves an ability to realize the equivalence of procedures which exist in different representations. Noting that the processes of symbolically solving $f(x) = 0$ and graphically finding x-intercepts are equivalent (in the sense of finding zeroes) demonstrates this awareness. Second, students develop the ability to generalize procedures across different classes and types of functions, particularly in algebra courses which introduce the elementary polynomials in a traditional sequence. Students at this second stage can translate procedures across representations (Stage 1), but are also beginning to realize that some of these procedures have analogues in other types of functions (such as finding x-intercepts of linear and quadratic polynomials). Third, once a student has become familiar with

various functional properties, he or she can "see" a function as an object either with or without these properties. For example, a quadratic function could be viewed as a continuous function with exactly one extrema, at most two zeroes, and which is symmetric about a vertical line (with, of course, second-degree growth). Just as a relational view is convenient when dealing with set-based operations on functions (such as composition), a growth-oriented conception helps to relate specific examples of functions to their corresponding growth behaviors.

The development of a growth-oriented conception of function will take time since it is dependent on knowledge of several different functional properties, representations, and classes of functions. If linear and quadratic functions are studied almost exclusively, as is the case in many high school Algebra I courses, then a student's "library" of functional properties will be quite small. To this student, functions are certainly well-behaved and continuous, have very simple growth behaviors, are either monotonic or change the direction of growth once, and are zero at most twice. When other polynomial (usually up to fourth- or fifth-degree), exponential and logarithmic, trigonometric, radical, and absolute value functions are added, as is the case in many high school Algebra II courses, the student's library of functional properties will increase, and a growth-oriented conception of function could deepen. By seeing examples of different types of functions that share functional properties (linear and simple radical functions sharing monotonicity, quadratic and simple absolute value functions sharing a unique extrema, quadratic and some trigonometric functions sharing symmetry), and by noting similarities and differences in the functions, the student can relate specific properties to a variety of exemplars. But not all radical functions are monotonic, nor are all trigonometric functions symmetric. Further delineations relative to specific function classes continually need to be made in order to classify functions in regard to functional properties, but this is an exercise that could only strengthen a growth-oriented view. This situation suggests that the question of the sufficiency of the kinds of functions that are currently most often studied in today's high schools be reexamined in regard to allowing students access to a broader array of functional properties. An obvious question concerns the adequacy of the functional properties normally encountered in regard to a student's development of the growth-oriented view. Perhaps other types of functions should be introduced to allow for a broader range of functional properties to be introduced. These could include discontinuities, finite domains, multivariable functions, or non-functional relations. One dimension on which this question should be addressed is the types of functions which give rise to contexts and situations that support investigation of algebraic and functional ideas. The connection of a functional property to a situational meaning can help strengthen an understanding of that property. It is also interesting to note that some functional properties, such as cusps and points of inflection, are not usually studied until ideas and techniques of calculus can help make them more explicit. This suggests the consideration of the role of function in advanced mathematics as another dimension to address the above question.

Synthesis of the Views

The previous views of function are not presented as disjoint avenues of student development, nor are they intended to completely describe all of the ways in which students can develop a concept image of function. Further, the above discussion of the growth-oriented conception is in line with characteristics of algebra courses which utilize multi-representational instructional approaches, as well as student responses to this instruction (Ruthven, 1990; Slavit, 1993, 1994; Teles, 1989). It must also be stated that the intent of this paper is not to develop a stage theory for the development of function. It is quite possible that a growth-oriented conception of function may be the first structural view that the student possesses, but it is also possible that it may be the last. Instruction is likely to influence the sequencing of this development.

Students who acquire a structural view can investigate functional contexts without a reliance on procedures and local function behaviors. Instead, emphasis can be placed on global behaviors such as growth rate or on relationships between specific local properties. The structural view is needed in order for a student to establish a proceptual understanding of functional notations (Gray and Tall, 1994), an understanding that transcends the action/object duality. Students who see functional notations proceptually can have the flexibility to think about function as an action, an object, or both. Further flexibility arises when the student is able to consider functions as mathematical objects in more than one way, such as the above relational and growth-oriented views. A structural view also allows the student to better understand actions performed on a function, such as a "shift" translation (e.g., changing $f(x)$ to $f(x + 3)$) or taking a derivative. It would be quite difficult for a student to completely understand an action he or she performed on a function if a structural view of function was not yet achieved.

Most importantly, we must remember the comments of Schoenfeld et al. (1993):

> Saying when a student actually "has" the object perspective is not a simple matter. It is not a yes/no kind of knowledge, but one of degrees, and the process of learning is not of simple monotonic growth, but one that includes a fair amount of oscillation (p. 88).

Empirical Support

Prior studies provide some empirical support for the theory. However, it should be clearly stated that a theory of a growth-oriented conception of function should be supported by more data before any curricular decisions are made which have the theory as a basis.

But there is evidence that students obtain a growth-oriented view of function, particularly when exposed to instruction that makes uses of graphs and graphing technologies. Ruthven (1990) found that students who used graphing calculators were better able to describe a given graph in symbolic terms, and that their ability

to do this relied heavily on their knowledge of functional properties. These students were also better able to identify and distinguish between classes of functions. Confrey and Smith (1995) argue that students are more likely to think of functions in terms of the covariation of two variables rather than in terms of the relational view which currently dominates the curriculum. However, they fall short of investigating student conceptions of functional properties as they relate to overall growth behavior, focusing only on general covariation relationships (e.g., in the linear equation y=2x+3, the established relationship could be "twice x plus three").

Slavit (1994) provides long-term data from a year-long study conducted in a high school Honors Algebra II course which made extensive use of graphing calculators. Questionnaires, test items, and case study interviews were recorded. Data indicated that some students reified function using growth-oriented notions. The strongest evidence came from an analysis of translation tasks, in which students used functional properties to relate different function representations, particularly when the graphic representation was involved. Further, because the instruction primarily focused on elementary polynomials, the students made false generalizations in regard to the properties which functions can possess, such as the need to be continuous. This led to misconceptions in their overall concept image. In addition, several examples of a growth-oriented view were found to "naturally" occur during problem solving episodes.

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